Journal of Approximation Theory **118**, 316–329 (2002)
doi:10.1006/jath.2002.3727

Rate of Convergence of the Linear Discrete Pólya 1-Algorithm[1]

J. M. Quesada[2], J. Martínez-Moreno, and J. Navas

Departamento de Matemáticas, Universidad de Jaén, Paraje las Lagunillas, Campus Universitario, 23071 Jaén, Spain
E-mail: jquesada@ujaen.es, jmmoreno@ujaen.es, jnavas@ujaen.es

and

J. Fernández-Ochoa

Departamento de Matemáticas, I.E.S. Albariza, Mengibar, 23620 Jaén, Spain

Communicated by Allan Pinkus

Received April 8, 2002; accepted in revised form June 26, 2002

In this paper, we consider the problem of best approximation in $\ell_p(n)$, $1 \leqslant p \leqslant \infty$. If h_p, $1 \leqslant p \leqslant \infty$, denotes the best ℓ_p-approximation of the element $h \in \mathbb{R}^n$ from a proper affine subspace K of $\mathbb{R}^n$, $h \notin K$, then $\lim_{p \to 1} h_p = h_1^*$, where h_1^* is a best ℓ_1-approximation of h from K, the so-called natural ℓ_1-approximation. Our aim is to give a complete description of the rate of convergence of h_p to h_1^* as $p \to 1$.

Key Words: best approximation; natural approximation; rate of convergence; Pólya algorithm.

1. INTRODUCTION

For $x = (x(1), x(2), \ldots, x(n)) \in \mathbb{R}^n$, the ℓ_p-norms, $1 \leqslant p \leqslant \infty$, are defined by

$$\|x\|_p = \left(\sum_{j=1}^{n} |x(j)|^p \right)^{1/p}, \qquad 1 \leqslant p < \infty,$$

$$\|x\|_\infty = \max_{1 \leqslant j \leqslant n} |x(j)|.$$

For convenience we will use $\|\cdot\|$ as the norm $\|\cdot\|_\infty$.

[1] This work was partially supported by Junta de Andalucía, Research Groups 0178, 0268 and by Ministerio de Ciencia y Tecnología, Project BFM2000-0911.
[2] To whom all correspondence should be addressed.

0021-9045/02 $35.00

Let $K \neq \emptyset$ be a subset of $\mathbb{R}^n$. For $h \in \mathbb{R}^n \backslash K$ and $1 \leqslant p \leqslant \infty$ we say that $h_p \in K$ is a best ℓ_p-approximation of h from K if

$$||h_p - h||_p \leqslant ||f - h||_p \qquad \text{for all } f \in K.$$

If K is a closed set of $\mathbb{R}^n$, then the existence of h_p is guaranteed. Moreover, there exists a unique best ℓ_p-approximation if K is a closed convex set and $1 < p < \infty$. In general, the unicity of the best ℓ_1-approximation is not guaranteed. Throughout this paper, K denotes a proper affine subspace of $\mathbb{R}^n$ and we will assume, without loss of generality, that $h = 0$ and $0 \notin K$. In this context, we gave in [5] a complete description of the rate of convergence of the Pólya algorithm as $p \to \infty$. We will apply similar techniques to study the case $p \to 1$.

It is known, [2, 3], that if K is an affine subspace of $\mathbb{R}^n$, then $\lim_{p \to 1} h_p = h_1^*$, where h_1^* is a best ℓ_1-approximation of 0 from K called the natural best ℓ_1-approximation. In the literature, the above convergence is called the Pólya 1-algorithm. The natural ℓ_1-approximation satisfies the following property. If L denotes the set of the best ℓ_1-approximations of 0 from K then h_1^* is the unique element of L that minimizes the expression

$$\sum_{j=1}^{n} |h_1(j)| \ln |h_1(j)|,$$

for all $h_1 \in L$, where $0 \ln 0 := 0$.

In [1] it is proved that $||h_p - h_1^*||/(p-1)$ is bounded as $p \to 1^+$. Also in [1], the authors show that if L is a singleton then $||h_p - h_1^*|| = \mathcal{O}(\gamma^{1/(p-1)})$ for some $0 \leqslant \gamma < 1$. However, the next example shows that this condition is not necessary to guarantee an exponential rate of convergence.

Example 1.1. Let us consider the affine subspace

$$K = (0, 1, 1) + \operatorname{span}\{(0, 1, -1), (2, 1, 0)\}.$$

If $h \in K$, then we can write $h = (2\mu, 1 + \lambda + \mu, 1 - \lambda)$, for some $\lambda, \mu \in \mathbb{R}$. Since, for all $\mu \in \mathbb{R} \backslash \{0\}$,

$$||h||_1 = |2\mu| + |1 + \lambda + \mu| + |1 - \lambda| > |1 + \lambda| + |1 - \lambda| \geqslant 2,$$

we conclude that the set of best ℓ_1-approximations of 0 from K is given by

$$L = \{(0, 1 + \lambda, 1 - \lambda) : |\lambda| \leqslant 1\}.$$

Moreover, the function $\Phi(\lambda) = (1 + \lambda)\ln(1 + \lambda) + (1 - \lambda)\ln(1 - \lambda)$, with $\lambda \in [-1, 1]$, has a minimum at $\lambda = 0$ and therefore $h_1^* = (0, 1, 1)$. On the other hand, an easy computation shows that for $p > 1$,

$$h_p = \left(\frac{-4}{1 + 2^{(2p-1)/(p-1)}}, 1 - \frac{1}{1 + 2^{(2p-1)/(p-1)}}, 1 - \frac{1}{1 + 2^{(2p-1)/(p-1)}} \right),$$

and we immediately deduce that $\lim_{p\to 1^+} 2^{1/(p-1)} ||h_p - h_1^*|| = 1$, and so, $||h_p - h_1^*|| = \mathcal{O}\left(\left(\frac{1}{2}\right)^{1/(p-1)} \right)$.

Our aim is to give a complete description of the rate of convergence of $||h_p - h_1^*||$ as $p \to 1^+$. More precisely, we establish necessary and sufficient conditions on K which guarantee that $||h_p - h_1^*||/(p-1)^k \to 0$ as $p \to 1^+$, for some $k \in \mathbb{N}$, and conditions on K for $h_p = h_1^*$ for all $p > 1$, and also conditions for when we obtain an exponential rate of convergence.

2. NOTATION AND PRELIMINARY RESULTS

Let $J := \{1, 2, \ldots, n\}$. For $x \in \mathbb{R}^n$, we define $Z(x) := \{j \in J : x(j) = 0\}$ and $R(x) := J \backslash Z(x)$. Let L be the set of best ℓ_1-approximations of 0 from K and let h_1^* be the natural best ℓ_1-approximation. We write $K = h_1^* + \mathscr{V}$, where $\mathscr{V}$ is a proper linear subspace of $\mathbb{R}^n$. For $v \in \mathscr{V}$, $v \neq 0$, we consider the function of the real variable $\lambda \in [0, +\infty)$ defined by

$$\varphi_v(\lambda) = \sum_{j \in J} |h_1^*(j) + \lambda\, v(j)| \ln |h_1^*(j) + \lambda v(j)|.$$

Observe that φ_v is a convex function if $0 \leqslant \lambda \leqslant \min_{j \in R(h_1^*) \cap R(v)} |h_1^*(j)|/|v(j)|$. Moreover, if $Z(h_1^*) \subseteq Z(v)$, then

$$\varphi_v'(0^+) = \sum_{R(h_1^*)} v(j)(1 + \ln |h_1^*(j)|) \operatorname{sgn}(h_1^*(j)), \tag{1}$$

otherwise $\varphi_v'(0^+) = -\infty$.

LEMMA 2.1. *If $h_1 \in L$, then $Z(h_1^*) \subseteq Z(h_1)$.*

Proof. Suppose that there exists an $h_1 \in L$ such that $h_1(j) \neq 0$ for some $j \in Z(h_1^*)$. If we take the vector $v = h_1 - h_1^* \neq 0$, then $\varphi_v'(0^+) = -\infty$ and so the function φ_v is strictly decreasing in $[0, \lambda_0]$ for some $\lambda_0 > 0$ sufficiently small. Then $\tilde{h}_1 = h_1^* + \lambda_0 v = (1 - \lambda_0) h_1^* + \lambda_0 h_1$ is in L and contradicts the definition of h_1^*. ∎

The following result is known (see for instance [4, 6]).

THEOREM 2.1 (Characterization of Best ℓ_p-Approximation). *h_p, $1 \leqslant p < \infty$, is a best ℓ_p-approximation of 0 from K if and only if for all $v \in \mathscr{V}$*

$$\sum_{j \in J} v(j)|h_p(j)|^{p-1}\,\mathrm{sgn}(h_p(j)) = 0 \qquad \text{if } 1 < p < \infty \tag{2}$$

or

$$\left|\sum_{R(h_1)} v(j)\,\mathrm{sgn}(h_1(j))\right| \leqslant \sum_{Z(h_1)} |v(j)| \qquad \text{if } p = 1. \tag{3}$$

To simplify the notation, we henceforth set $J_0 = Z(h_1^*)$ and $J_0^c = R(h_1^*)$. Let $v \in \mathscr{V}$, $v \neq 0$. For $\lambda > 0$ sufficiently small, we have

$$\begin{aligned}
||h_1^* + \lambda v||_1 &= \sum_{j \in J} |h_1^*(j) + \lambda\, v(j)| \\
&= \sum_{j \in J_0^c} (h_1^*(j) + \lambda v(j))\,\mathrm{sgn}(h_1^*(j)) + \lambda \sum_{j \in J_0} |v(j)| \\
&= \sum_{j \in J_0^c} |h_1^*(j)| + \lambda\left(\sum_{j \in J_0^c} v(j)\,\mathrm{sgn}(h_1^*(j)) + \sum_{j \in J_0} |v(j)|\right) \\
&= ||h_1^*||_1 + \lambda\left(\sum_{j \in J_0^c} v(j)\,\mathrm{sgn}(h_1^*(j)) + \sum_{j \in J_0} |v(j)|\right).
\end{aligned}$$

From the above equality we immediately obtain the following result.

LEMMA 2.2. *Let $v \in \mathscr{V}$, $v \neq 0$. Then $\tilde{h} = h_1^* + \lambda\, v$, with $\lambda > 0$ sufficiently small, is a best ℓ_1-approximation of 0 from K if and only if*

$$\sum_{j \in J_0^c} v(j)\,\mathrm{sgn}(h_1^*(j)) + \sum_{j \in J_0} |v(j)| = 0.$$

COROLLARY 2.1. *Let $v \in \mathscr{V}$, $v \neq 0$. If $J_0 \subseteq Z(v)$ then*

(a) $\sum_{j \in J_0^c} v(j)\,\mathrm{sgn}(h_1^*(j)) = 0$,

(b) $h_1^* + \lambda v \in L$, *for λ sufficiently small,*

(c) $\sum_{j \in J_0^c} v(j) \ln |h_1^*(j)|\,\mathrm{sgn}(h_1^*(j)) = 0$, *otherwise,*

$$\left|\sum_{j \in J_0^c} v(j)\,\mathrm{sgn}(h_1^*(j))\right| < \sum_{j \in J_0} |v(j)|. \tag{4}$$

Proof. If $J_0 \subseteq Z(v)$, then conditions (a) and (b) follow immediately from (3) and Lemma 2.2. Now from (a) and (1) we have

$$\varphi_v'(0^+) = \sum_{j \in J_0^c} v(j) \ln|h_1^*(j)| \operatorname{sgn}(h_1^*(j)). \tag{5}$$

Since $h_1^* + \lambda v \in L$ for λ sufficiently small, the definition of h_1^* implies that $\varphi_v'(0^+) \geqslant 0$. Replacing v by $-v$ in (5) we conclude (c). On the other hand, if $J_0 \not\subset Z(v)$, then the strict inequality in (4) follows from (3) and Lemmas 2.1 and 2.2. ∎

COROLLARY 2.2 (Pinkus [4, p. 135]). *h_1^* is the unique best ℓ_1-approximation of* 0 *from K if and only if* (4) *holds for all $v \in \mathscr{V}$, $v \neq 0$.*

3. RATE OF CONVERGENCE

Let $\beta \neq 0$ and $\{\alpha_p\}$ be a sequence of real numbers such that $\alpha_p \to 0$ as $p \to 1^+$. Applying the Mean Value Theorem to the function $f(t) = |\beta + t|^{p-1}$ it may be shown that

$$|\beta + \alpha_p|^{p-1} = |\beta|^{p-1} + (p-1)\,\alpha_p\,\eta_p, \tag{6}$$

where $\lim_{p \to 1^+} \eta_p = 1/\beta$. Now, applying Taylor's formula of order r to the function $g(p) = |\beta|^{p-1}$ at $p = 1$, we can write

$$|\beta + \alpha_p|^{p-1} = \sum_{k=0}^{r} \frac{\ln^k|\beta|}{k!}(p-1)^k + (p-1)\alpha_p\eta_p + \mathcal{O}((p-1)^{r+1}). \tag{7}$$

Let $\mathscr{V}_0$ be the linear subspace of $\mathscr{V}$ defined by $\mathscr{V}_0 := \{u \in \mathscr{V} : u(j) = 0,\ \forall j \in J_0\}$. Note that if $J_0 = \emptyset$, then $\mathscr{V}_0 = \mathscr{V}$.

LEMMA 3.1. *Suppose that there exists a $v \in \mathscr{V}_0$ and $r \in \mathbb{N}$ such that*

$$\sum_{j \in J_0^c} v(j) \ln^k|h_1^*(j)| \operatorname{sgn}(h_1^*(j)) = 0, \qquad 0 \leqslant k \leqslant r. \tag{8}$$

Then

$$\lim_{p \to 1^+} \frac{1}{(p-1)^r} \sum_{j \in J_0^c} v(j) \frac{h_p(j) - h_1^*(j)}{|h_1^*(j)|} = -\frac{1}{(r+1)!} \sum_{j \in J_0^c} v(j) \ln^{r+1}|h_1^*(j)| \operatorname{sgn}(h_1^*(j)). \tag{9}$$

Proof. Let $v \in \mathscr{V}_0$. From (2) we have, for p close to 1,

$$\sum_{j \in J_0^c} v(j)|h_p(j)|^{p-1} \operatorname{sgn}(h_1^*(j)) = 0. \tag{10}$$

Since $||h_p - h_1^*|| \to 0$ as $p \to 1^+$, applying (7) with r replaced by $r+1$, we get for every $j \in J_0^c$

$$\begin{aligned} |h_p(j)|^{p-1} &= |h_1^*(j) + h_p(j) - h_1^*(j)|^{p-1} \\ &= \sum_{k=0}^{r+1} \frac{\ln^k |h_1^*(j)|}{k!}(p-1)^k + (p-1)(h_p(j) - h_1^*(j))\eta_p + \mathcal{O}((p-1)^{r+2}), \end{aligned}$$

where $\lim_{p \to 1^+} \eta_p(j) = 1/h_1^*(j)$. Placing this expression in (10) and taking into account (8) we obtain

$$\begin{aligned} &\frac{(p-1)^{r+1}}{(r+1)!} \sum_{j \in J_0^c} v(j) \ln^{r+1} |h_1^*(j)| \operatorname{sgn}(h_1^*(j)) \\ &\quad + (p-1) \sum_{j \in J_0^c} v(j)(h_p(j) - h_1^*(j))|\eta_p(j)| + \mathcal{O}((p-1)^{r+2}) = 0. \end{aligned}$$

Now, dividing by $(p-1)^{r+1}$ and letting $p \to 1^+$ we obtain (9). ∎

Observe that, from Corollary 2.1, (8) holds for all $v \in \mathscr{V}_0$ and $r = 1$. So we can establish the following result.

COROLLARY 3.1. *For all* $v \in \mathscr{V}_0$,

$$\lim_{p \to 1^+} \frac{1}{p-1} \sum_{j \in J_0^c} v(j) \frac{h_p(j) - h_1^*(j)}{|h_1^*(j)|} = -\frac{1}{2} \sum_{j \in J_0^c} v(j) \ln^2 |h_1^*(j)| \operatorname{sgn}(h_1^*(j)).$$

Now, we can give a very easy proof of the main result in [1].

THEOREM 3.1 (Egger and Taylor [1, Theorem 2]). *There exists a constant* $M > 0$ *such that* $||h_p - h_1^*|| \leqslant M(p-1)$ *for all* $p > 1$.

Proof. If the assertion is false, then there exists a sequence $\{p_k\}_{k \in \mathbb{N}}$ such that $p_k \downarrow 1$ and $||h_{p_k} - h_1^*||/(p_k - 1) \to \infty$ as $k \to \infty$. Thus, we will prove the theorem by showing that for any sequence $p_k \downarrow 1$, $\liminf ||h_{p_k} - h_1^*||/(p_k - 1) < \infty$. So let $p_k \downarrow 1$. If $h_{p_k} = h_1^*$ for infinite many k, then the result follows. Using a subsequence if necessary, we may therefore suppose that

$h_{p_k} \neq h_1^*$ for all k and $\lim_{k\to\infty} u_k = u \in \mathscr{V}$, with $||u|| = 1$, where

$$u_k = \frac{h_{p_k} - h_1^*}{||h_{p_k} - h_1^*||}.$$

We consider two cases.

(a) If $u \in \mathscr{V}_0$, then, from Corollary 3.1, we have

$$\lim_{k\to\infty} \frac{1}{p_k - 1} \sum_{j\in J_0^c} u(j) \frac{h_{p_k}(j) - h_1^*(j)}{|h_1^*(j)|} = -\frac{1}{2} \sum_{j\in J_0^c} u(j) \ln^2|h_1^*(j)| \operatorname{sgn}(h_1^*(j)),$$

hence

$$\lim_{k\to\infty} \frac{||h_{p_k} - h_1^*||}{p_k - 1} \sum_{j\in J_0^c} \frac{u(j)u_k(j)}{|h_1^*(j)|} = -\frac{1}{2} \sum_{j\in J_0^c} u(j) \ln^2|h_1^*(j)| \operatorname{sgn}(h_1^*(j))$$

and

$$\lim_{k\to\infty} \frac{||h_{p_k} - h_1^*||}{p_k - 1} = \frac{|\sum_{j\in J_0^c} u(j) \ln^2|h_1^*(j)| \operatorname{sgn}(h_1^*(j))|}{2 \sum_{j\in J_0^c} |u(j)|^2/|h_1^*(j)|}.$$

In particular, we have obtained that $||h_{p_k} - h_1^*||/(p_k - 1)$ is bounded.

(b) If $u(j) \neq 0$ for some $j \in J_0$, then, applying (2), we have, for k large,

$$\sum_{j\in J_0^c} u(j)|h_{p_k}(j)|^{p_k-1} \operatorname{sgn}(h_1^*(j)) + \sum_{j\in J_0} |u(j)||h_{p_k}(j)|^{p_k-1} = 0, \tag{11}$$

and so,

$$\sum_{j\in J_0^c} u(j)|h_{p_k}(j)|^{p_k-1} \operatorname{sgn}(h_1^*(j)) + ||h_{p_k} - h_1^*||^{p_k-1} \sum_{j\in J_0} |u(j)||u_k(j)|^{p_k-1} = 0.$$

Thus, by Corollary 2.1

$$\lim_{k\to\infty} ||h_{p_k} - h_1^*||^{p_k-1} = \frac{|\sum_{j\in J_0^c} u(j) \operatorname{sgn}(h_1^*(j))|}{\sum_{j\in J_0} |u(j)|} = \gamma_u < 1.$$

Therefore, if $0 < \gamma_u < \gamma < 1$ then there exists a $k_0 \in \mathbb{N}$, such that

$$||h_{p_k} - h_1^*|| < \gamma^{1/(p_k-1)}, \qquad \text{for all } k > k_0.$$

In particular, $||h_{p_k} - h_1^*||/(p_k - 1)$ is bounded. ∎

Note that in case (b) of Theorem 3.1 we have obtained a exponential rate of convergence of h_p to h_1^*. In the next results, we will examine necessary and sufficient conditions which guarantee this rate.

For $J' \subseteq J$, we will denote by $||\cdot||_{J'}$ the restriction of $||\cdot||$ to the set of indices in J'.

Lemma 3.2. *If* $||h_p - h_1^*||/(p-1)^r$ *is bounded for some* $r \in \mathbb{N}$, *then* $||h_p||_{J_0}/(p-1)^r \to 0$ *as* $p \to 1^+$. *In particular*, $||h_p||_{J_0}/(p-1) \to 0$ *as* $p \to 1^+$.

Proof. Assume that the claim is false. Since $||h_p - h_1^*||/(p-1)^r$ is bounded, we can take a subsequence $p_k \to 1^+$ such that $u_k := (h_{p_k} - h_1^*)/(p_k - 1)^r \to u \in \mathscr{V}$ and $u(j) \neq 0$ for some $j \in J_0$. By (2) we obtain, for k large,

$$\sum_{j \in J_0^c} u(j)|h_{p_k}(j)|^{p_k - 1} \operatorname{sgn}(h_1^*(j)) + \sum_{j \in J_0} |u(j)||h_{p_k}(j)|^{p_k - 1} = 0$$

or equivalently

$$\sum_{j \in J_0^c} u(j)|h_{p_k}(j)|^{p_k - 1} \operatorname{sgn}(h_1^*(j)) + (p_k - 1)^{r(p_k - 1)} \sum_{j \in J_0} |u(j)||u_k(j)|^{p_k - 1} = 0.$$

Letting $k \to \infty$, we have

$$\sum_{j \in J_0^c} u(j) \operatorname{sgn}(h_1^*(j)) + \sum_{j \in J_0} |u(j)| = 0,$$

which is a contradiction by Lemmas 2.1 and 2.2. The second assertion follows from Theorem 3.1. ∎

The following result establishes necessary and sufficient conditions in order to obtain a rate of convergence faster than $p-1$.

Corollary 3.2. $\lim_{p \to 1} ||h_p - h_1^*||/(p-1) \to 0$ *if and only if*

$$\sum_{j \in J_0^c} v(j) \ln^2 |h_1^*(j)| \operatorname{sgn}(h_1^*(j)) = 0 \qquad \forall v \in \mathscr{V}_0. \tag{12}$$

Proof. Assume that there exists a $v \in \mathscr{V}_0$ such that (12) does not hold. Taking into account that

$$\left| \sum_{j \in J_0^c} v(j) \frac{h_p(j) - h_1^*(j)}{|h_1^*(j)|} \right| \leqslant ||h_p - h_1^*|| \sum_{j \in J_0^c} |v(j)|/|h_1^*(j)|,$$

we deduce, from Corollary 3.1, that there exists an $M > 0$ and a $p_0 > 1$ such that $||h_p - h_1^*|| > M(p-1)$, $1<p<p_0$. We thus conclude that $||h_p - h_1^*||/(p-1)$ does not converge to 0 as $p \to 1$. Conversely, suppose that (12) holds for all $u \in \mathscr{V}_0$ and that $||h_p - h_1^*||/(p-1)$ does not converge to 0 as $p \to 1^+$. Let $u \in \mathscr{V}$ be the vector defined as in Lemma 3.2. Then $u \in \mathscr{V}_0$ and $u(j) \neq 0$ for some $j \in J_0^c$. From Corollary 3.1 we obtain

$$\sum_{j \in J_0^c} u(j)^2/|h_1^*(j)| = 0.$$

A contradiction. ∎

Using a similar argument as above and Lemma 3.2 we can establish the following general result.

COROLLARY 3.3. *Let* $r \in \mathbb{N}$. *Then* $\lim_{p\to\infty} ||h_p - h_1^*||/(p-1)^r \to 0$ *if and only if*

$$\sum_{j \in J_0^c} v(j) \ln^k |h_1^*(j)| \operatorname{sgn}(h_1^*(j)) = 0, \qquad 1 \leqslant k \leqslant r+1,$$

for all $v \in \mathscr{V}_0$.

Next, we study necessary and sufficient conditions needed to obtain an exponential rate of convergence. From Corollaries 2.1 and 3.3 it will be necessary that for all $v \in \mathscr{V}_0$,

$$\sum_{j \in J_0^c} v(j) \ln^k |h_1^*(j)| \operatorname{sgn}(h_1^*(j)) = 0 \qquad \text{for all } k \in \mathbb{N} \cup \{0\}. \tag{13}$$

Henceforth, we use the following notation. Let $0 < d_1 < d_2 < \cdots < d_s$ be all the different values of $|h_1^*(j)|$ on J_0^c. We consider the partition $\{J_l\}_{l=1}^s$ of J_0^c into sets defined by $J_l = \{j \in J_0^c : |h_1^*(j)| = d_l\}$.

Condition (13) may be replaced by another condition which is easier to verify.

LEMMA 3.3. *Let* $v \in \mathscr{V}$. *The following conditions are equivalent*:

(i) $\sum_{j \in J_0^c} v(j) \ln^k |h_1^*(j)| \operatorname{sgn}(h_1^*(j)) = 0$ *for all* $k \in \mathbb{N} \cup \{0\}$.

(ii) $\sum_{j \in J_0^c} v(j) \ln^k |h_1^*(j)| \operatorname{sgn}(h_1^*(j)) = 0$, $\quad 0 \leqslant k \leqslant s-1$.

(iii) $\sum_{j \in J_l} v(j) \operatorname{sgn}(h_1^*(j)) = 0$, $\quad 1 \leqslant l \leqslant s$.

(iv) $\sum_{j \in J_0^c} v(j)|h_1^*(j)|^{p-1} \operatorname{sgn}(h_1^*(j)) = 0$ *for all* $p > 1$.

(v) *There exist* $1 < p_1 < p_2 < \cdots < p_s$ *such that*

$$\sum_{j\in J_0^c} v(j)|h_1^*(j)|^{p_k-1}\,\mathrm{sgn}(h_1^*(j)) = 0, \qquad 1 \leqslant k \leqslant s. \tag{14}$$

Proof. (i) $\Rightarrow$ (ii): It is obvious.

(ii) $\Rightarrow$ (iii): Putting $\rho_l = \ln(d_l)$, $1 \leqslant l \leqslant s$, we can write

$$\sum_{j\in J_0^c} v(j)\ln^k |h_1^*(j)|\,\mathrm{sgn}(h_1^*(j)) = \sum_{l=1}^{s} \rho_l^k \sum_{j\in J_l} v(j)\,\mathrm{sgn}(h_1^*(j)) = 0.$$

Replacing $k = 0, 1, \ldots, s-1$, we obtain an $s \times s$ nonsingular homogeneous linear system of unknowns $\sum_{j\in J_l} v(j)\,\mathrm{sgn}(h_1^*(j))$, $1 \leqslant l \leqslant s$ (observe that the determinant of the matrix of coefficients is a Vandermonde determinant). Then we deduce that

$$\sum_{j\in J_l} v(j)\,\mathrm{sgn}(h_1^*(j)) = 0, \qquad 1 \leqslant j \leqslant s.$$

(iii) $\Rightarrow$ (iv): For all $p > 1$ we have

$$\sum_{j\in J_0^c} v(j)|h_1^*(j)|^{p-1}\,\mathrm{sgn}(h_1^*(j)) = \sum_{l=1}^{s} d_l^{p-1} \sum_{j\in J_l} v(j)\,\mathrm{sgn}(h_1^*(j)) = 0.$$

(iv) $\Rightarrow$ (v): It is immediate.

(iv) $\Rightarrow$ (i): It follows immediately by calculating the derivatives with respect to p of order $k = 0, 1, \ldots,$ of the expression

$$\sum_{j\in J_0^c} v(j)|h_1^*(j)|^{p-1}\,\mathrm{sgn}(h_1^*(j)) = 0$$

and letting $p \to 1^+$.

Finally, we prove (v) $\Rightarrow$ (ii). We use an argument similar to that in (i) $\Rightarrow$ (ii). Indeed, writing

$$\begin{aligned}&\sum_{j\in J_0^c} v(j)|h_1^*(j)|^{p_k-1}\,\mathrm{sgn}(h_1^*(j))\\ &\quad= \sum_{l=1}^{s} d_l^{p_l-1} \sum_{j\in J_l} v(j)\,\mathrm{sgn}(h_1^*(j)) = 0, \qquad 1 \leqslant l \leqslant s,\end{aligned}$$

we also obtain an $s \times s$ nonsingular homogeneous linear system of equations (in this case the determinant of the matrix of coefficients is a generalized Vandermonde determinant). ∎

COROLLARY 3.4. *$h_p = h_1^*$ for all $p > 1$ if and only if*

$$\sum_{j \in J_l} v(j)\,\mathrm{sgn}(h_1^*(j)) = 0, \qquad 1 \leqslant l \leqslant s$$

for all $v \in \mathscr{V}$.

Proof. From (2) we have $h_p = h_1^*$ if and only if

$$\sum_{j \in J_0^c} v(j)|h_1^*(j)|^{p-1}\,\mathrm{sgn}(h_1^*(j)) = 0, \qquad \forall v \in \mathscr{V}.$$

Applying Lemma 3.3 we conclude the proof. ∎

Let $\mathscr{B}_0$ be a basis of $\mathscr{V}_0$, where we will assume that $\mathscr{B}_0 := \emptyset$ if $\mathscr{V}_0 = \{0\}$. Let $\mathscr{B}_1$ be a set (possibly empty) of linearly independent vectors in $\mathscr{V}$ such that $\mathscr{B} := \mathscr{B}_0 \cup \mathscr{B}_1$ is a basis of $\mathscr{V}$ and let $\mathscr{V}_1 = \mathrm{span}(\mathscr{B}_1)$. Then we can write $\mathscr{V} = \mathscr{V}_0 \oplus \mathscr{V}_1$.

THEOREM 3.2. *Suppose that*

$$\sum_{j \in J_l} v(j)\,\mathrm{sgn}(h_1^*(j)) = 0, \qquad 1 \leqslant l \leqslant s, \tag{15}$$

for all $v \in \mathscr{B}_0$. If (15) *also holds for all $v \in \mathscr{B}_1$, then $h_p = h_1^*$ for all $p > 1$. Otherwise, let*

$$\gamma_0 := \max_{\substack{\|v\|=1 \\ v \in \mathscr{V}_1}} \frac{|\sum_{j \in J_0^c} v(j)\,\mathrm{sgn}(h_1^*(j))|}{\sum_{j \in J_0} |v(j)|} < 1.$$

Then for all $\gamma_0 < \gamma < 1$ there are $p_0 = p_0(\gamma) > 1$ and an $M > 0$ such that $\|h_p - h_1^*\| < M\gamma^{1/(p-1)}$ for all $1 < p < p_0$.

Proof. The first assertion is a consequence of Corollary 3.4. Suppose there exists a $\tilde{v} \in \mathscr{B}_1$ such that (15) does not hold. By Corollary 3.4 and Lemma 3.3, this implies that there exists a $p_1 > 1$ such that $h_p \neq h_1^*$ for $1 < p < p_1$. Put $h_p = h_1^* + w_p^{(0)} + w_p^{(1)}$, with $w_p^{(0)} \in \mathscr{V}_0$ and $w_p^{(1)} \in \mathscr{V}_1$. First, we prove that $w_p^{(1)} \neq 0$ for p close to 1. To the contrary, $h_p(j) = 0$, all $j \in J_0$, and from (2) we obtain for p close to 1,

$$\sum_{j\in J_0^c} \tilde{v}(j)|h_p(j)|^{p-1}\operatorname{sgn}(h_1^*(j)) = 0. \tag{16}$$

Let $r \in \mathbb{N}$ arbitrary. By (7), we can write,

$$\begin{aligned}|h_p(j)|^{p-1} = |h_1^*(j) + w_p^{(0)}(j)|^{p-1} &= \sum_{k=0}^{r} \frac{\ln^k |h_1^*(j)|}{k!}(p-1)^k \\ &+ (p-1)w_p^{(0)}(j)\eta_p(j) + \mathcal{O}(p-1)^{r+1},\end{aligned} \tag{17}$$

where $\lim_{p\to 1^+} \eta_p(j) = 1/|h_1^*(j)|$. From the hypothesis and Corollary 3.3, we have $\lim_{p\to 1^+}||w_p^{(0)}||/(p-1)^k = \lim_{p\to 1^+}||h_p - h_1^*||/(p-1)^k = 0$, for all $k \in \mathbb{N}$. So replacing (17) in (16), dividing by $(p-1)^k$, $0 \leqslant k \leqslant r$ and taking limits as $p \to 1$ we conclude that

$$\sum_{j\in J_0^c} \tilde{v}(j)\ln^k|h_1^*(j)|\operatorname{sgn}(h_1^*(j)) = 0, \qquad 0\leqslant k\leqslant r.$$

Since the above equality holds for all $r \in \mathbb{N}$, Lemma 3.3 implies that $\tilde{v}$ satisfies (15). This is a contradiction. We therefore conclude that $w_p^{(1)} \neq 0$ for p close to 1. Taking a subsequence, if necessary, we consider the unit vector $w \in \mathscr{V}_1$ given by $w = \lim_{p\to 1^+} w_p^{(1)}/||w_p^{(1)}||$. Applying (2) we have, for p close to 1,

$$\sum_{j\in J_0^c} w(j)|h_p(j)|^{p-1}\operatorname{sgn}(h_1^*(j)) + \sum_{j\in \hat{J}_0} w(j)|h_p(j)|^{p-1}\operatorname{sgn}(h_p(j)) = 0,$$

where $\hat{J}_0 := \{j \in J_0 : w(j) \neq 0\} \neq \emptyset$. Since, $h_p(j) = w_p^{(1)}(j)$ if $j \in \hat{J}_0$, we can rewrite the above equation as

$$\sum_{j\in J_0^c} w(j)|h_p(j)|^{p-1}\operatorname{sgn}(h_1^*(j)) + ||w_p^{(1)}||^{p-1}\sum_{j\in \hat{J}_0} |w(j)|\left|\frac{w_p^{(1)}(j)}{||w_p^{(1)}||}\right|^{p-1} = 0.$$

So

$$\lim_{p\to 1^+} ||w_p^{(1)}||^{p-1} = \frac{|\sum_{j\in J_0^c} w(j)\operatorname{sgn}(h_1^*(j))|}{\sum_{j\in J_0} |w(j)|} = \gamma_w \leqslant \gamma_0 < 1,$$

and hence for $\gamma_w < \gamma < 1$ there exists a $p_2 = p_2(\gamma) > 1$ such that $||w_p^{(1)}|| < \gamma^{1/(p-1)}$ for $1 < p < p_2$. If $\mathscr{B}_0 = \emptyset$ or $w_p^{(0)} = 0$ all $p > 1$, then the proof is complete. Otherwise, using a subsequence if necessary, we define the unit vector $\tilde{u} \in \mathscr{V}$ given by $\tilde{u} = \lim_{p\to 1^+} w_p^{(0)}/||w_p^{(0)}||$. By (2) we have for p

close to 1

$$\sum_{j\in J_0^c} \tilde{u}(j)\,|h_p(j)|^{p-1}\,\mathrm{sgn}(h_1^*(j)) = 0.$$

By (6) we can write

$$\begin{aligned}|h_p(j)|^{p-1} &= |h_1^*(j) + w_p^{(0)}(j) + w_p^{(1)}(j)|^{p-1}\\ &= |h_1^*(j)|^{p-1} + (p-1)(w_p^{(0)}(j) + w_p^{(1)}(j))\eta_p(j),\end{aligned}$$

where $\lim_{p\to 1^+}\eta_p(j) = 1/h_1^*(j)$. Placing this in the previous equation we obtain

$$\sum_{j\in J_0^c} \tilde{u}(j)|h_1^*(j)|^{p-1}\,\mathrm{sgn}(h_1^*(j)) + (p-1)\sum_{j\in J_0^c} \tilde{u}(j)(w_p^{(0)}(j) + w_p^{(1)}(j))|\eta_p(j)| = 0.$$

From the hypothesis and Lemma 3.3, we have

$$\sum_{j\in J_0^c} \tilde{u}(j)|h_1^*(j)|^{p-1}\,\mathrm{sgn}(h_1^*(j)) = 0$$

and so

$$||w_p^{(0)}||\sum_{j\in J_0^c} \tilde{u}(j)\,\frac{w_p^{(0)}(j)}{||w_p^{(0)}||}\,|\eta_p(j)| + \sum_{j\in J_0^c} \tilde{u}(j)\,w_p^{(1)}(j)\,|\eta_p(j)| = 0.$$

It is now easy to deduce that there exists an $M>0$ and $1<p_0\leqslant p_2$ such that $||w_p^{(0)}||\leqslant M\gamma^{1/(p-1)}$ for $1<p<p_0$. This conclude the proof. ∎

Observe that if h_1^* is the unique best ℓ_1-approximation of 0 from K, then Corollary 2.2 implies that $\mathscr{B}_0=\emptyset$ and so (15) holds obviously.

COROLLARY 3.5. *If L is a singleton then there exists a γ, $0<\gamma<1$, such that $||h_p - h_1^*||<\gamma^{1/(p-1)}$.*

The following table summarizes the previous results and gives a complete description of the rate of convergence of h_p to h_1^* as $p\to 1^+$. Given $v\in\mathscr{V}$ we denote, for short,

$$\Sigma^k_{J_0^c}(v) = \sum_{j\in J_0^c} v(j)\ln^k|h_1^*(j)|\,\mathrm{sgn}(h_1^*(j)),\qquad \Sigma_{J_l}(v) = \sum_{j\in J_l} v(j)\,\mathrm{sgn}(h_1^*(j))$$

and we consider

$$r_0 = \max\Big\{r \in \{1, \dots, s-1\} : \ \Sigma^k_{J_0^c}(v) = 0, \ 0 \leqslant k \leqslant r, \ \forall v \in \mathscr{B}_0\Big\}.$$

Condition on $\mathscr{V}$	Rate
$\Sigma_{J_l}(v) = 0, \quad 1 \leqslant l \leqslant s, \qquad \forall v \in \mathscr{B}_0 \cup \mathscr{B}_1$	$h_p = h_1^*$, for all $p > 1$
$\Sigma_{J_l}(v) = 0, \quad 1 \leqslant l \leqslant s, \qquad \forall v \in \mathscr{B}_0$ $\Sigma_{J_l}(v) \neq 0$, for some $1 \leqslant l \leqslant s$ and some $v \in \mathscr{B}_1$	$\mathcal{O}(\gamma^{1/(p-1)})$
$r_0 < s-1$	$\mathcal{O}((p-1)^{r_0})$

Note that, from Corollary 2.1, $r_0 \geqslant 1$. On the other hand, from Lemma 3.3, $r_0 = s-1$ is equivalent to $\Sigma_{J_l}(v) = 0$, $1 \leqslant l \leqslant s$, for all $v \in \mathscr{B}_0$.

ACKNOWLEDGMENTS

The authors gratefully acknowledge the many helpful suggestions of the referees during the revision of this paper.

REFERENCES

1. A. G. Egger and G. D. Taylor, Rate of convergence of the discrete Pólya-1 algorithm, *J. Approx. Theory* **75** (1993), 312–324.
2. J. Fischer, The convergence of the best discrete linear L_p approximation as $p \to 1$, *J. Approx. Theory* **39** (1983), 374–385.
3. D. Landers and L. Rogge, Natural choice of L_1-approximants, *J. Approx. Theory* **33** (1981), 268–280.
4. A. Pinkus, "On L_1-Approximation," Cambridge Univ. Press, Cambridge, UK, 1989.
5. J. M. Quesada and J. Navas, Rate of convergence of the linear discrete Polya algorithm, *J. Approx. Theory* **110** (2001), 109–119.
6. I. Singer, "Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces," Springer-Verlag, Berlin, 1970.